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# High Order Accurate Weighted Essentially Non-oscillatory Algorithms for Shock Calculations

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## 1. Summary of research

We have performed research on the algorithm development, analysis, implementation and application for high order weighted essentially non-oscillatory (WENO) finite difference and finite volume schemes and related methods, for solving computational fluid dynamics (CFD) problems and other applications containing strong shock waves and complicated smooth region structures. The goal of this research is to help obtaining more robust, cost effective, and reliable numerical tools for computational fluid dynamics problems and other applications, that are of interest to AFOSR.

There are 22 publications in refereed journals, which have appeared or have been accepted for publications and have quoted partial support by this grant, see section 2.

In [11] (the numbers here and below refer to that of publications listed in section 2), we have developed a fifth order weighted essentially non-oscillatory (WENO) fast sweeping method for solving static Hamilton-Jacobi equations, utilizing two approaches of accurate inflow boundary condition treatment which allows the usage of Cartesian meshes regardless of the domain boundary shape. This method is expected to very useful in a wide range of applications involving level set methods and front propagation.

In [1], we have developed high order essentially non-oscillatory (ENO) Lagrangian schemes based on the Lax-Wendroff (LW) type time discretization procedure. Extensive numerical examples are presented, for both the second order spatial discretization using quadrilateral meshes and third order spatial discretization using curvilinear meshes. These methods are expected to be very useful for multi-phase flows and they can capture phase boundaries sharply.

In [15], we have developed a framework to construct uniformly high order accurate, total variation diminishing finite volume and discontinuous Galerkin schemes

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13. SUPPLEMENTARY NOTES

14. ABSTRACT
**We have performed research on the algorithm development, analysis, implementation and application for high order weighted essentially non-oscillatory (WENO) finite difference and finite volume schemes and related methods, for solving computational fluid dynamics (CFD) problems and other applications containing strong shock waves and complicated smooth region structures. The goal of this research is to help obtaining more robust, cost effective, and reliable numerical tools for computational fluid dynamics problems and other applications, that are of interest to AFOSR. 22 papers in positivity-preserving high order schemes, stable and accurate boundary conditions for high order finite difference methods, efficient semi-Lagrangian finite difference schemes and other topics are published in refereed journals.**

15. SUBJECT TERMS

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for solving one dimensional nonlinear scalar conservation laws. The main idea is to measure the total variation of the complete piecewise polynomial, to design a simple scalar limiter which keeps high order accuracy, and to follow exact time evolution. The method in this paper overcomes the traditional barriers of second order accuracy for total variation diminishing schemes. However, the requirement of exact time evolution prevents its generalization to multi-dimensions and systems. In [16], we have adapted and generalized this framework to construct uniformly high order accurate finite volume and discontinuous Galerkin schemes satisfying a strict maximum principle for scalar conservation laws. The main breakthrough includes a framework allowing the maximum principle for the cell averages with Euler forward time discretization without affecting spatial high order accuracy, and a simple scaling limiter which requires the evaluation of the polynomial solution only at fixed quadrature points and keeps high order accuracy. This breakthrough allows us to design high order schemes satisfying a strict maximum principle for multi-dimensional scalar problems and passive convection in a divergence-free velocity field, including two-dimensional incompressible Euler equations in vorticity-streamfunction formulation. The methodology is extended in [17] to Euler systems of compressible gas dynamics to maintain positivity for density and pressure without affecting high order accuracy, and in [18] to compressible Euler equations with geometric, reaction, gravity and other types of source terms. An adaptation of the methodology to high order finite difference WENO schemes is carried out in [20], and a minimum entropy principle of high order schemes for gas dynamics equations in [21]. Finally, generalizations to arbitrary unstructured triangular meshes are carried out in [22]. This line of research has led to the first genuinely uniformly high order schemes which satisfy strict maximum principle and/or positivity-preserving properties as appropriate, which are very simple to implement and have shown excellent performance in very demanding numerical tests such as detonation waves in [8]. We expect this methodology to be widely used in applications. An extension to convection diffusion equations for a non-standard finite volume scheme is established in [14]. The student co-author, Dr. Xiangxiong Zhang (who obtained his Ph.D. degree in 2011), has been chosen as one of the three winners in the prestigious SIAM Student Paper Competition for his joint paper [16].

In [5], we have developed a high order finite difference numerical boundary condition for solving hyperbolic conservation laws on a Cartesian mesh. The challenge results from the wide stencil of the interior high order scheme and the fact that the physical boundary may not be aligned with the grid points in the Cartesian mesh and may intersect the grids in an arbitrary fashion. Our method is based on an inverse Lax-Wendroff procedure for the inflow boundary conditions. Extensive numerical examples are provided to illustrate that our method is high order accurate and has good performance when applied to one and two dimensional scalar or system cases with the physical boundary not aligned with the grids and with various boundary conditions including the solid wall boundary condition. In [6], this technique is generalized to compressible inviscid flows involving complex moving geometries. Numerical examples in one and two dimensions show that our boundary treatment is high order accurate for problems with smooth solutions. Our method also performs well for problems involving interactions between shocks and moving rigid bodies. In [7], a more efficient implementation of this technique is introduced and the scheme is applied to very demanding test cases in detonation wave propagation. The methodology in this line of work is expected to significantly expand the applicability of high order finite difference scheme on Cartesian meshes for solving problems in complex geometry with stable and accurate boundary performance.

In [2,3], we have developed several types of conservative high order semi-Lagrangian finite difference WENO methods for advection in incompressible flow with applications to the Vlasov equation. Each type has its own advantages and limitations, however collectively these methods can address many problems in plasma physics very efficiently. In [4], we have developed a positivity-preserving semi-Lagrangian discontinuous Galerkin with applications to Vlasov-Poisson systems.

In [9], we have discussed the importance of well-balanced schemes for moving-water equilibria of the shallow water equations.

In [10], we have developed a WENO scheme with subcell resolution for computing nonconservative Euler equations with applications to one-dimensional compressible two-medium flows. This is a first step towards the development of a robust and accurate solver for multi-medium flows.

In [12], we have systematically studied a high order adaptive finite element method for solving nonlinear hyperbolic conservation laws.

In [13], we have studied a strategy to improve convergence to steady state solutions of Euler equations with high order WENO finite difference schemes. This is expected to improve significantly the efficiency of high order finite difference WENO solvers for steady state flows.

## 2. Publications in refereed journals (appeared or accepted) which have quoted partial support by this grant


1. W. Liu, J. Cheng and C.-W. Shu, *High order conservative Lagrangian schemes with Lax-Wendroff type time discretization for the compressible Euler equations*, Journal of Computational Physics, v228 (2009), pp.8872-8891.

2. J.-M. Qiu and C.-W. Shu, *Conservative high order semi-Lagrangian finite difference WENO methods for advection in incompressible flow*, Journal of Computational Physics, v230 (2011), pp.863-889.

3. J.-M. Qiu and C.-W. Shu, *Conservative semi-Lagrangian finite difference WENO formulations with applications to the Vlasov equation*, Communications in Computational Physics, v10 (2011), pp.979-1000.

4. J.-M. Qiu and C.-W. Shu, *Positivity preserving semi-Lagrangian discontinuous Galerkin formulation: theoretical analysis and application to the Vlasov-Poisson system*, Journal of Computational Physics, v230 (2011), pp.8386-8409.

5. S. Tan and C.-W. Shu, *Inverse Lax-Wendroff procedure for numerical boundary conditions of conservation laws*, Journal of Computational Physics, v229 (2010), pp.8144-8166.

6. S. Tan and C.-W. Shu, *A high order moving boundary treatment for compressible inviscid flows*, Journal of Computational Physics, v230 (2011), pp.6023-6036.

7. S. Tan, C. Wang, C.-W. Shu and J. Ning, *Efficient implementation of high order inverse Lax-Wendroff boundary treatment for conservation laws*, Journal of Computational Physics, to appear.

8. C. Wang, X. Zhang, C.-W. Shu and J. Ning, *Robust high order discontinuous Galerkin schemes for two-dimensional gaseous detonations*, Journal of Computational Physics, v231 (2012), pp.653-665.

9. Y. Xing, C.-W. Shu and S. Noelle, *On the advantage of well-balanced schemes for moving-water equilibria of the shallow water equations*, Journal of Scientific Computing, v48 (2011), pp.339-349.

10. T. Xiong, C.-W. Shu and M. Zhang, *WENO scheme with subcell resolution for computing nonconservative Euler equations with applications to one-dimensional compressible two-medium flows*, Journal of Scientific Computing, to appear.

11. T. Xiong, M. Zhang, Y.-T. Zhang and C.-W. Shu, *Fifth order fast sweeping WENO scheme for static Hamilton-Jacobi equations with accurate boundary treatment*, Journal of Scientific Computing, v45 (2010), pp.514-536.

12. Z. Xu, J. Xu and C.-W. Shu, *A high order adaptive finite element method for solving nonlinear hyperbolic conservation laws*, Journal of Computational Mathematics, v29 (2011), pp.491-500.

13. S. Zhang, S. Jiang and C.-W. Shu, *Improvement of convergence to steady state solutions of Euler equations with the WENO schemes*, Journal of Scientific Computing, v47 (2011), pp.216-238.

14. X. Zhang, Y.-Y. Liu and C.-W. Shu, *Maximum-principle-satisfying high order finite volume WENO schemes for convection-diffusion equations*, SIAM Journal on Scientific Computing, to appear.

15. X. Zhang and C.-W. Shu, *A genuinely high order total variation diminishing scheme for one-dimensional scalar conservation laws*, SIAM Journal on Numerical Analysis, v48 (2010), pp.772-795.

16. X. Zhang and C.-W. Shu, *On maximum-principle-satisfying high order schemes for scalar conservation laws*, Journal of Computational Physics, v229 (2010), pp.3091-3120.

17. X. Zhang and C.-W. Shu, *On positivity preserving high order discontinuous Galerkin schemes for compressible Euler equations on rectangular meshes*, Journal of Computational Physics, v229 (2010), pp.8918-8934.

18. X. Zhang and C.-W. Shu, *Positivity-preserving high order discontinuous Galerkin schemes for compressible Euler equations with source terms*, Journal of Computational Physics, v230 (2011), pp.1238-1248.

19. X. Zhang and C.-W. Shu, *Maximum-principle-satisfying and positivity-preserving high order schemes for conservation laws: Survey and new developments*, Proceedings of the Royal Society A, v467 (2011), pp.2752-2776.

20. X. Zhang and C.-W. Shu, *Positivity-preserving high order finite difference WENO schemes for compressible Euler equations*, Journal of Computational Physics, v231 (2012), pp.2245-2258.

21. X. Zhang and C.-W. Shu, *A minimum entropy principle of high order schemes for gas dynamics equations*, Numerische Mathematik, to appear.

22. X. Zhang, Y. Xia and C.-W. Shu, *Maximum-principle-satisfying and positivity-preserving high order discontinuous Galerkin schemes for conservation laws on triangular meshes*, Journal of Scientific Computing, v50 (2012), pp.29-62.